Mathematical Morphology

- Set-theoretic representation for binary shapes

Qigong Zheng
Language and Media Processing Lab
Center for Automation Research
University of Maryland College Park
October 31, 2000

What is the mathematical morphology ?

- An approach for processing digital image based on its **shape**
- A mathematical tool for investigating **geometric structure** in image
- The language of morphology is **set theory**

Goal of morphological operations

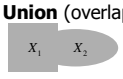
- Simplify image data, preserve essential shape characteristics and eliminate noise
- Permits the underlying shape to be identified and optimally reconstructed from their distorted, noisy forms

Shape Processing and Analysis

- Identification of objects, object features and assembly defects correlate directly with **shape**
- Shape is a prime carrier of information in machine vision

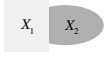
Shape Operators

- Shapes are usually combined by means of :
  - **Set Union** (overlapping objects):



$X_1 \cup X_2$

  - **Set Intersection** (occluded objects):



$X_2 \setminus X_1 = X_1^c \cap X_2$

Morphological Operations

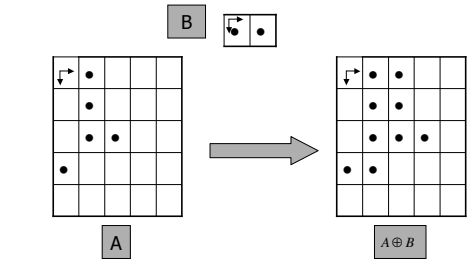
- The primary morphological operations are **dilation** and **erosion**
- More complicated morphological operators can be designed by means of combining erosions and dilations

Dilation

- Dilation is the operation that combines two sets using vector addition of set elements.
- Let A and B are subsets in 2-D space. A: image undergoing analysis, B: Structuring element, $\oplus$ denotes dilation

$$A \oplus B = \{c \in Z^2 \mid c = a + b \ \ for \ \ some \ \ a \in A, b \in B\}$$

Dilation



Dilation

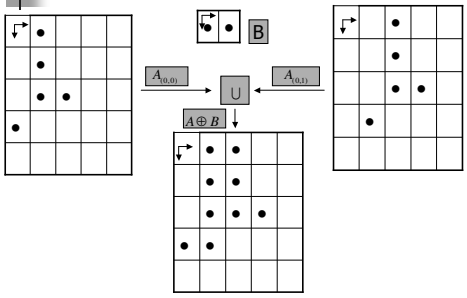
- Let A be a Subset of Z^2 and $x \in Z^2$. The translation of A by x is defined as

$$(A)_x = \{c \in Z^2 \mid c = a + x, \ \ for \ \ some \ \ a \in A\}$$

- The dilation of A by B can be computed as the union of translation of A by the elements of B

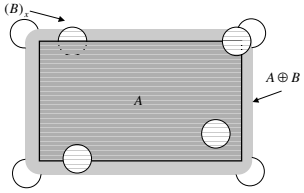
$$A \oplus B = \bigcup_{b \in B} (A)_b = \bigcup_{a \in A} (B)_a$$

Dilation

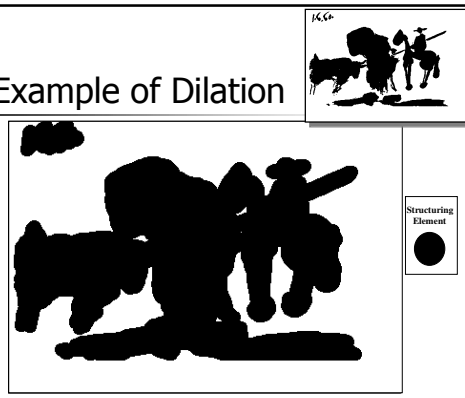


Dilation

$$A \oplus B = \bigcup_{b \in B} (A)_b = \bigcup_{a \in A} (B)_a$$



Example of Dilation



Properties of Dilation

- Commutative

$$A \oplus B = B \oplus A$$

- Associative

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

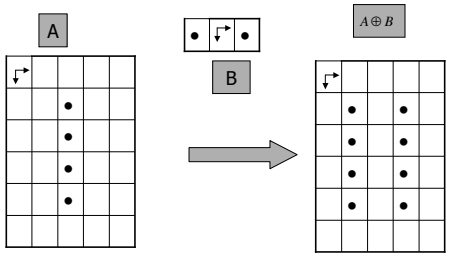
- Extensivity

$$if \;\; 0 \in B, A \subseteq A \oplus B$$

- Dilation is increasing

$$A \subseteq B \; implies \; A \oplus D \subseteq B \oplus D$$

Extensitivity



Properties of Dilation

- Translation Invariance

$$(A)_x \oplus B = (A \oplus B)_x$$

- Linearity

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$$

- Containment

$$(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$$

- Decomposition of structuring element

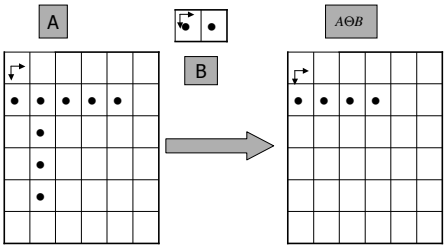
$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

Erosion

- Erosion is the morphological dual to dilation. It combines two sets using the vector subtraction of set elements.
- Let $A \Theta B$ denotes the erosion of A by B

$$A \Theta B = \{x \in Z^2 \mid \; for \; every \; b \in B, exist \; an \; a \in A \; s.t. x = a - b\}$$
$$= \{x \in Z^2 \mid x + b \in A \; \; for \; every \; b \in B)$$

Erosion



Erosion

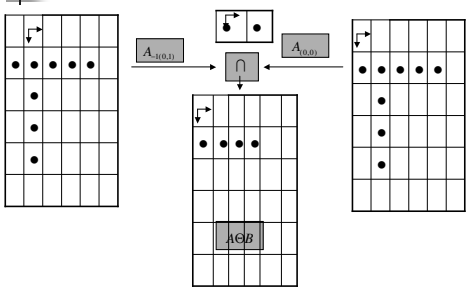
- Erosion can also be defined in terms of translation

$$A \Theta B = \{x \in Z^2 \mid (B)_x \subseteq A)$$

- In terms of intersection

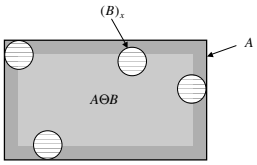
$$A \Theta B = \bigcap_{b \in B} (A)_{-b}$$

Erosion

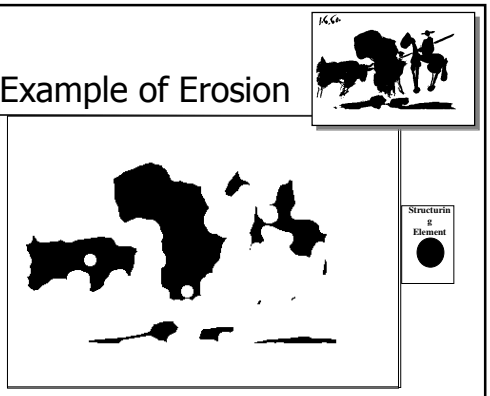


Erosion

$$A\Theta B = \{x \in Z^2 | (B)_x \subseteq A)$$



Example of Erosion



Properties of Erosion

- Erosion is not commutative!

$$A\Theta B \neq B\Theta A$$

- Extensivity

$$if \ \ 0 \in B, A\Theta B \subseteq A$$

- Dilation is increasing

$$A \subseteq C \ implies \ A\Theta B \subseteq C\Theta B, \ B \supseteq C \ implies \ A\Theta B \subseteq A\Theta C$$

- Chain rule

$$A\Theta(B_1 \oplus ... \oplus B_k) = (...(A\Theta B_1)\Theta ...\Theta B_k)$$

Properties of Erosion

- Translation Invariance

$$A_x\Theta B = (A\Theta B)_x, \ A\Theta B_x = (A\Theta B)_{-x}$$

- Linearity

$$(A \cap B)\Theta C = (A\Theta C) \cap (B\Theta C)$$

- Containment

$$(A \cup B)\Theta C \supseteq (A\Theta C) \cup (B\Theta C)$$

- Decomposition of structuring element

$$A\Theta(B \cup C) = (A\Theta B) \cap (A\Theta C)$$

Duality Relationship

- Dilation and Erosion transformation bear a marked similarity, in that what one does to image foreground and the other does for the image background.
- $B \in Z^2$, the reflection of B, $\bar{B}$ is defined as

$$\bar{B} = \{x | \ for \ some \ b \in B, \ x = -b\}$$

- Erosion and Dilation Duality Theorem

$$(A \Theta B)^c = A^c \oplus \bar{B}$$

Opening and Closing

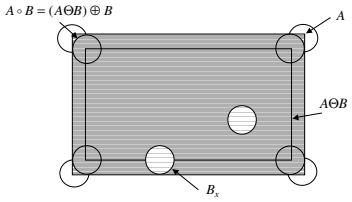
- Opening and closing are iteratively applied dilation and erosion

Opening $A \circ B = (A \ominus B) \oplus B$

Closing $A \bullet B = (A \oplus B) \ominus B$

Opening and Closing

$$A \circ B = \bigcup_{\{x \mid B_x \subseteq A\}} B_x$$



Opening and Closing

- They are idempotent. Their reapplication has not further effects to the previously transformed result

$$A \bullet B = (A \bullet B) \bullet B$$

$$A \circ B = (A \circ B) \circ B$$

Opening and Closing

- Translation invariance

$$A \circ (B)_x = A \circ B \qquad A \bullet (B)_x = A \bullet B$$

- Antiextensivity of opening

$$A \circ B \subseteq A$$

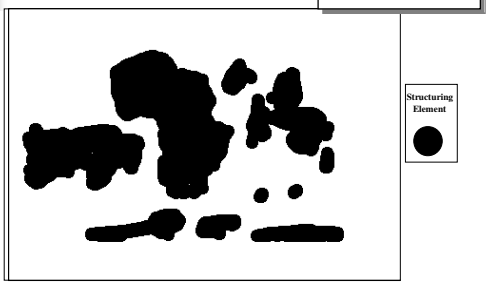
- Extensivity of closing

$$A \subseteq A \bullet B$$

- Duality

$$(A \bullet B)^c = A^c \circ \check{B}$$

Example of Opening



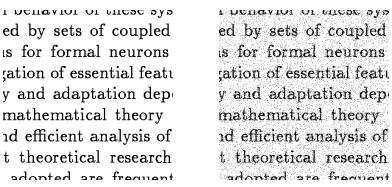
Example of Closing

Morphological Filtering

- Main idea
  - Examine the **geometrical** structure of an image by matching it with small patterns called structuring elements at various locations
  - By varying the **size** and **shape** of the matching patterns, we can extract useful information about the shape of the different parts of the image and their interrelations.

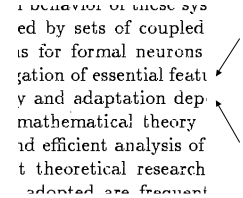
Morphological filtering

- Noisy image will break down OCR systems



Clean image | Noisy image

Morphological filtering



By applying MF, we increase the OCR accuracy!

Restored image

Summary

- Mathematical morphology is an approach for processing digital image based on its **shape**
- The language of morphology is **set theory**
- The basic morphological operations are **erosion** and **dilation**
- Morphological filtering can be developed to extract useful shape information

THE END