3-Dimensional Rotation and Quaternions

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Specifying placements in 3-space

Specifying the *placement* of an object or camera in 3-space requires two things:

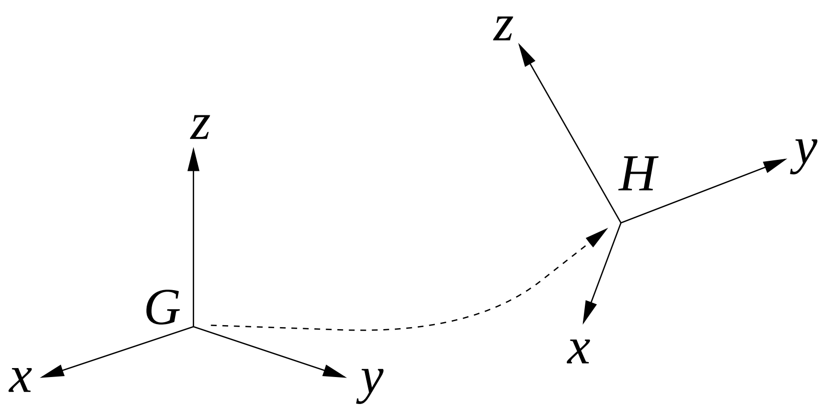
Location: The coordinates of the objects reference point.

Orientation: The rotation of the object with respect to its reference point.

Location is easy to understand, represent, and interpolate with. Rotation in 3-space is much trickier.

The Basic Problem

Given two orthonormal frames, G and H, determine a smooth transformation (*interpolation function*) from one to the other.

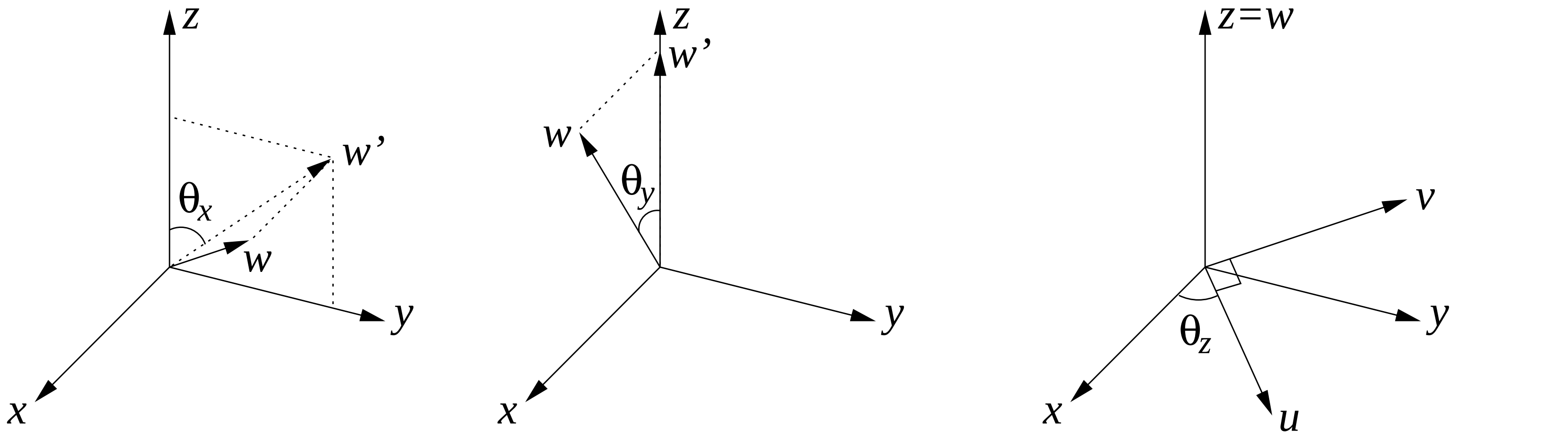


We will discuss two methods: *Euler angles* and *quaternions*.

Coordinate Rotations

Given any two orthonormal frames in 3-space sharing the same origin, Euler showed that one can be mapped to the other by three rotations (called *rolls*) about the coordinate axes.

It suffices to map a frame $\langle \vec{u}, \vec{v}, \vec{w} \rangle$ to the standard frame.



(a) Rotate about x so that w lies in xz-plane.

(b) Rotate about y so that w coincides with z.

(c) Rotate about z so that u and v coincide with x and y.

Euler Angles and Interpolation

It is possible to perform any change of orientation with three rotations about the coordinate axes.

$$R(\theta_x, \theta_y, \theta_z) = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x).$$

These three angles are called the *Euler angles* for the rotation.

Now, given two orientations in space, by their Euler angles we can interpolate between them:

$$R(\alpha) = R((1-\alpha)\Theta + \alpha\Phi),$$

for $0 \leq \alpha \leq 1$.

Rotation Matrices

We can map Euler angles to an equivalent matrix form (assuming homogeneous column vectors) by composing these rotations:

$$\begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & -\sin\theta_y & 0 \\ 0 & \sin\theta_y & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_x & 0 & \sin\theta_x & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_x & 0 & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Rotation Matrix

Let $c_x = \cos\theta_x$, $s_x = \sin\theta_x$, and so on. Then

$$R(\theta_x, \theta_y, \theta_z) = \begin{pmatrix} c_y c_z & c_y s_z & -s_y & 0 \\ s_x s_y c_z - c_x s_z & s_x s_y s_z + c_x c_z & s_x c_y & 0 \\ c_x s_y c_z + s_x s_z & c_x s_y s_z - s_x c_z & c_x c_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note the absence of symmetry with respect to x, y, and z. The choice of axes and order of rotation are significant to the result.

Gimbal Lock

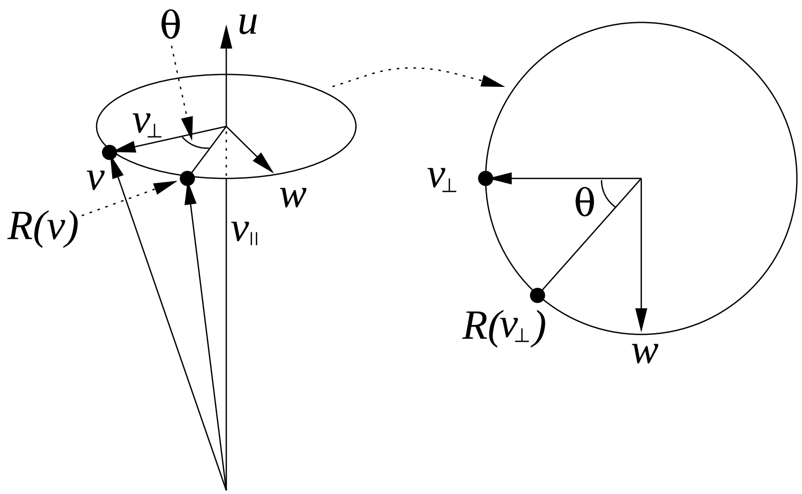
Under certain conditions a degree of freedom is lost, resulting in *gimbal lock*. Consider $R(\theta_x, \pi/2, \theta_z)$:

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ s_x c_z - c_x s_z & s_x s_z + c_x c_z & 0 & 0 \\ c_x c_z + s_x s_z & c_x s_z - s_x c_z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ \sin(\theta_x - \theta_z) & \cos(\theta_x - \theta_z) & 0 & 0 \\ \cos(\theta_x - \theta_z) & -\sin(\theta_x - \theta_z) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Angular Displacement

Let $R(\vec{v})$ denote rotation of $\vec{v}$ by angle θ about unit vector $\vec{u}$. Let us decompose $\vec{v}$ into its parallel and orthogonal components.

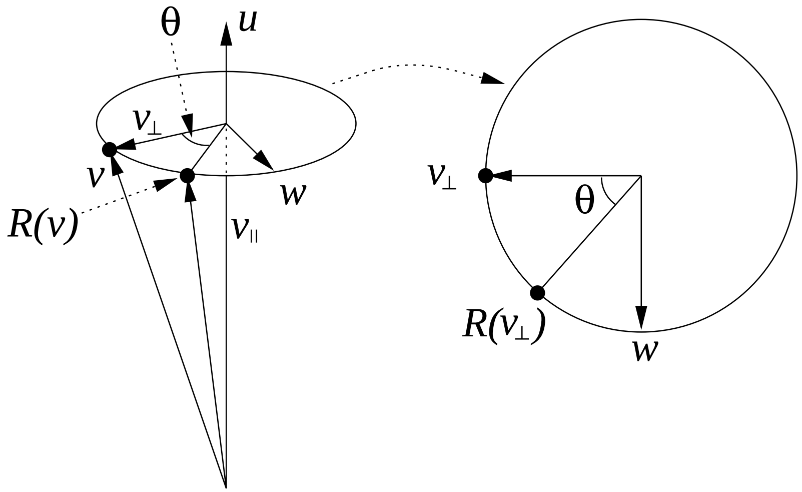
$$\vec{v}_{\|} = (\vec{u} \cdot \vec{v})\vec{u} \qquad \vec{v}_{\perp} = \vec{v} - \vec{v}_{\|} = \vec{v} - (\vec{u} \cdot \vec{v})\vec{u}.$$



Angular Displacement (cont)

Note that $\vec{v}_{\parallel}$ is unaffected by the rotation, but $\vec{v}_{\perp}$ is rotated to a new position $R(\vec{v}_{\perp})$. To determine this rotation, we construct a vector w that is orthogonal to $\vec{v}_{\perp}$ lying in the plane of rotation.

$$\vec{w} = \vec{u} \times \vec{v}_{\perp} = \vec{u} \times (\vec{v} - \vec{v}_{\parallel}) = (\vec{u} \times \vec{v}) - (\vec{u} \times \vec{v}_{\parallel}) = \vec{u} \times \vec{v}.$$



Angular Displacement (cont)

Now, consider the plane spanned by $\vec{v}_\perp$ and $\vec{w}$. We have

$$R(\vec{v}_\perp) = (\cos\theta)\vec{v}_\perp + (\sin\theta)\vec{w}.$$

From this we have

$$\begin{aligned} R(\vec{v}) &= R(\vec{v}_{||}) + R(\vec{v}_\perp) \\ &= R(\vec{v}_{||}) + (\cos\theta)\vec{v}_\perp + (\sin\theta)\vec{w} \\ &= (\vec{u}\cdot\vec{v})\vec{u} + (\cos\theta)(\vec{v} - (\vec{u}\cdot\vec{v})\vec{u}) + (\sin\theta)\vec{w} \\ &= (\cos\theta)\vec{v} + (1-\cos\theta)\vec{u}(\vec{u}\cdot\vec{v}) + (\sin\theta)(\vec{u}\times\vec{v}). \end{aligned}$$

Remember this formula. It will show up later.

Quaternions

In the 19th century, Sir William Hamilton was seeking a generalization of *complex numbers*. Given two unit convex numbers, multiplication is equivalent to rotation. For example:

$$\begin{aligned}(\cos\theta + i\sin\theta)^2 &= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) \\ &= \cos 2\theta + i\sin 2\theta.\end{aligned}$$

He was searching for a generalization to 3-space, involving two imaginary quantities i and j.

Quaternions

After 10 years of searching, he hit upon the idea of using three imaginary values i, j, and k, where

$$i^2 = j^2 = k^2 = -1 \qquad ij = k,\ jk = i,\ ki = j.$$

It follows that

$$ji = -k, \quad kj = -i \quad \text{and} \quad ik = -j.$$

A *quaternion* is defined to be a "generalized complex number" of the form

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k.$$

Quaternion Multiplication

Consider two quaternions $\mathbf{q}_1 = (s, \vec{u})$ and $\mathbf{q}_2 = (t, \vec{v})$. It is easy to verify the following.

$$\mathbf{q}_1 \mathbf{q}_2 = (st - (\vec{u} \cdot \vec{v}), s\vec{v} + t\vec{u} + (\vec{u} \times \vec{v})).$$

If you ignore the cross-product term, this bears a striking superficial resemblance to the rule for complex multiplication. Note that quaternion multiplication is associative but not commutative.

Other Quaternion Concepts

Conjugate: Of a quaternion $\mathbf{q} = (s, \vec{u})$:

$$\overline{\mathbf{q}} = (s, -\vec{u}).$$

Magnitude: It is easy to show that the product of a quaternion and its conjugate has a zero vector component, and hence may be thought of as a scalar.

$$|\mathbf{q}|^2 = \mathbf{q}\overline{\mathbf{q}} = s^2 + |\vec{u}|^2.$$

Other Quaternion Concepts

Unit quaternion: is one of unit magnitude.

Pure quaternion: is one with a 0 scalar component.

Inverse: Any quaternion of nonzero magnitude has a multiplicative inverse.

$$\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|^2}\overline{\mathbf{q}}.$$

If $\mathbf{q}$ is a unit quaternion, then $\mathbf{q}^{-1} = \overline{\mathbf{q}}$.

Quaternions and Rotation

We will represent a rotation by a unit quaternion $\mathbf{q}$. Given any point $P = (v_x, v_y, v_z)$ in 3-space, we will represent it by mapping it to a pure quaternion, $\mathbf{p} = (0, \vec{v})$. The image of $\mathbf{p}$ under the rotation $\mathbf{q}$ will again be a pure quaternion. Define the *rotation operator*

$$R_{\mathbf{q}}(\mathbf{p}) = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}.$$

By applying the multiplication rule, and using the fact that $\mathbf{q}^{-1} = \overline{\mathbf{q}}$ for unit quaternions, we have.

$$R_{\mathbf{q}}(\mathbf{p}) = (0,\ (s^2 - (\vec{u} \cdot \vec{u}))\vec{v} + 2\vec{u}(\vec{u} \cdot \vec{v}) + 2s(\vec{u} \times \vec{v})).$$

Quaternions and Rotation

So what does this have to do with rotation? Since $\mathbf{q}$ is of unit magnitude, we can express it as

$$\mathbf{q} = (\cos\theta, (\sin\theta)\vec{u}), \qquad \text{where } |\vec{u}| = 1.$$

Plugging this into the previous expression, we have

$$\begin{aligned} R_{\mathbf{q}}(\mathbf{p}) &= (0,\ (\cos^2\theta - \sin^2\theta)\vec{v} + 2(\sin^2\theta)\vec{u}(\vec{u}\cdot\vec{v}) + 2\cos\theta\sin\theta(\vec{u}\times\vec{v})) \\ &= (0,\ (\cos 2\theta)\vec{v} + (1-\cos 2\theta)\vec{u}(\vec{u}\cdot\vec{v}) + \sin 2\theta(\vec{u}\times\vec{v})). \end{aligned}$$

This is identical to the angular displacement formula given earlier, except that we have 2θ in place of θ.

Quaternions and Rotation

In summary, we encode points in 3-space as *pure quaternions*:

$$\mathbf{p} = (0, \vec{v}),$$

We encode a rotation by angle θ about a unit vector $\vec{u}$ as a unit quaternion:

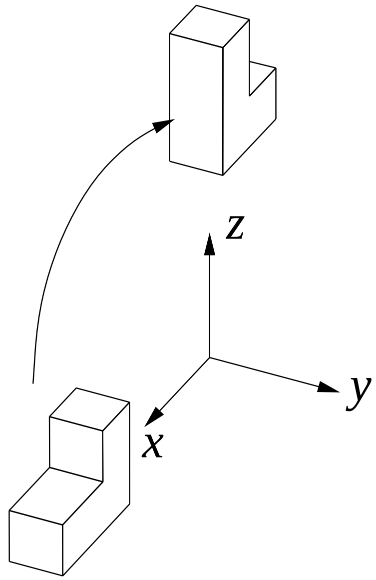
$$\mathbf{q} = (\cos(\theta/2), \sin(\theta/2)\vec{u}),$$

The image of the point under this rotation is given by the vector part of the result of the quaternion rotation operator $R_{\mathbf{q}}(\mathbf{p}) = \mathbf{qpq}^{-1}$.

Example

Consider a rotation about the y-axis by -90 degrees. Thus $\theta = -\pi/2$, and $\vec{u} = (0, 1, 0)$. Thus the quaternion that encodes this rotation is

$$\mathbf{q} = (\cos(-\pi/4), \sin(-\pi/4)(0,1,0)) = \left(\frac{1}{\sqrt{2}}, \left(0, -\frac{1}{\sqrt{2}}, 0\right)\right).$$



Example (cont)

The image of the point with homogeneous coordinates $(1, 0, 0, 1)$, comes by representing P by a vector $\vec{v} = (1, 0, 0)$, and then encoding $\vec{v}$ as a pure quaternion $\mathbf{p} = (0, (1, 0, 0))$. Then we apply the rotation operator

$$
\begin{aligned}
R_{\mathbf{q}}(\mathbf{p}) &= (0,\ (1/2 - 1/2)(1, 0, 0) + 2(0, 1, 0)0 \\
&\quad +\ (2/\sqrt{2})((0, -1/\sqrt{2}, 0) \times (1, 0, 0))) \\
&= (0, (0, 0, 0) + (0, 0, 0) + (-1)(0, 0, -1)) \\
&= (0, (0, 0, 1)).
\end{aligned}
$$

Thus P is mapped to a point on the z-axis, as expected.

Composing Rotations

We can manipulate rotations through quaternion operations. Multiplying two unit quaternions results in another unit quaternion, which corresponds to the composition of the two rotations. Consider two unit quaternions $\mathbf{q}$ and $\mathbf{q}'$. Let their product be $\mathbf{q}'' = \mathbf{q}'\mathbf{q}$.

$$\begin{aligned} R_{\mathbf{q}'}(R_{\mathbf{q}}(\mathbf{p})) &= \mathbf{q}'(\mathbf{q}\mathbf{p}\mathbf{q}^{-1})\mathbf{q}'^{-1} \\ &= (\mathbf{q}'\mathbf{q})\mathbf{p}(\mathbf{q}^{-1}\mathbf{q}'^{-1}) \\ &= (\mathbf{q}'\mathbf{q})\mathbf{p}(\mathbf{q}'\mathbf{q})^{-1} \\ &= \mathbf{q}''\mathbf{p}\mathbf{q}''^{-1} \\ &= R_{\mathbf{q}''}(\mathbf{p}). \end{aligned}$$

This follows from the associativity of quaternion multiplication, and the fact that $(\mathbf{q}\mathbf{q}')^{-1} = \mathbf{q}^{-1}\mathbf{q}'^{-1}$.

Matrix Representation

Can we convert a quaternion into a matrix? Given a unit quaternion

$$\mathbf{q} \;=\; (\cos(\theta/2), \sin(\theta/2)\vec{u}) \;=\; (w, (x, y, z)).$$

By simply expanding the definition of of $R_{\mathbf{q}}(\mathbf{p})$, it is not hard to show that the following (homogeneous) matrix is equivalent

$$\begin{pmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy & 0 \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx & 0 \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that unlike the Euler angle transformation matrix, this one is symmetric with respect to x, y, and z. The inverse process is also easy.

Interpolation

Our initial problem was smoothly interpolating between two given orientations. Given quaternions $\mathbf{q}_0$ and $\mathbf{q}_1$, we could use:

$$\mathbf{q}(t) = (1 - t)\mathbf{q}_0 + t\mathbf{q}_1$$

where $0 \leq t \leq 1$.

But the intermediate quaternions would not be unit quaternions. We need to interpolation along the sphere.

Slerping

We need to interpolate along a 4-dimensional sphere, called *spherical linear interpolation* (or *slerp*).

Consider the simpler problem of interpolating between two unit vectors, $\vec{u}_0$ and $\vec{u}_1$, along an arc of the unit circle. Let ϕ denote the angle between these vectors, and let $\theta = \phi t$ be the intermediate angle of the interpolated unit vector. We seek an interpolation of the form:

$$\vec{u}(t) = \alpha(t)\vec{u}_0 + \beta(t)\vec{u}_1.$$

for scalars $\alpha(t)$ and $\beta(t)$ (functions of t).

Slerping (cont)

We have

$$\vec{u}(t) = \alpha(t)\vec{u}_0 + \beta(t)\vec{u}_1.$$

and the constraints

$$(\vec{u}(t) \cdot \vec{u}(t)) = 1, \qquad (\vec{u}_0 \cdot \vec{u}_1) = \cos\phi, \qquad (\vec{u}_0 \cdot \vec{u}(t)) = \cos\theta.$$

Solving for $\alpha(t)$ and $\beta(t)$ we have

$$\begin{aligned} \vec{u}(t) &= \frac{\sin(\phi - \theta)}{\sin\phi}\vec{u}_0 + \frac{\sin\theta}{\sin\phi}\vec{u}_1 \\ &= \frac{\sin((1-t)\phi)}{\sin\phi}\vec{u}_0 + \frac{\sin t\phi}{\sin\phi}\vec{u}_1. \end{aligned}$$

The quaternion slerp is defined analogously

$$\mathbf{q}(t) = \frac{\sin((1-t)\phi)}{\sin\phi}\mathbf{q}_0 + \frac{\sin t\phi}{\sin\phi}\mathbf{q}_1.$$

Conclusions

Euler angles: So-so.

Quaternions: Better.