Morphological Decomposition of Restricted Domains: A Vector Space Solution*

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Abstract

We define a *restricted domains*, a class of discrete, 2D, convex shapes. We prove that there is a set of thirteen restricted domains $\{K_1, K_2, \ldots, K_{13}\}$ such that any given restricted domain, K, is expressible as $K = K_0 \oplus (\oplus_{k_1} K_1) \oplus (\oplus_{k_2} K_2) \oplus \cdots \oplus (\oplus_{k_{13}} K_{13})$ where $(\oplus_{k_i} K_i)$ represents the k_i-fold dilation of K_i and K_0 is a translation. We show that this entails a linear transformation from a thirteen dimensional space in which restricted domains are represented in terms of n-fold dilations of the thirteen basis structuring elements, to an eight dimensional space in which restricted domains are represented in terms of their eight side lengths. Furthermore, we show that *any* particular decomposition is a particular solution of this transformation and finding *all* possible dilation decompositions of a restricted domain is equivalent to finding the general solution of this transformation. Finally, we give an algorithm for finding all possible decompositions.

1 Introduction

Shape decomposition is important from the hardware point of view, [1, 5], as well as from the shape description point of view, [1]. In this paper we interpret the decomposition of restricted domains as a vector space problem. We show that the solution to the problem is not unique. Next we show that the decompositions obtained in [4, 5] are particular solutions of the linear transformation problem. We show that the solution of the decomposition problem is the sum of a particular solution and the homogeneous solutions and give algorithms for finding them. Part of the results presented here have been presented in [1, 2, 3, 4].

2 Restricted Domains

A 4 or 8 connected component A is *discretely convex* if and only if all the lattice points lying inside the convex hull of A belong to A. A *restricted domain* is a discretely convex, 4-connected shape whose convex hull has sides at angles which are multiples of 45 degrees with respect to the positive x axis.

Any restricted domain A can be represented by specifying its lowest and leftmost boundary point (i, j), and the lengths of the sides encountered while traversing the boundary of A in the counter-clockwise direction. The side lengths can be represented by a 8×1 vector $\mathbf{N} = [n_0\ n_1\ \cdots\ n_7]'$ whose components n_i, $0 \le i \le 7$, represent the lengths along the chain-code directions i. If there is no displacement corresponding to one of the directions i, $0 \le i \le 7$, the corresponding n_i is given a value of zero. See figure 1.

Figure 1: (a) Chain-code directions; (b) a restricted domain with $(i, j) = (1, 2)$ and $\mathbf{N} = [2\ 3\ 6\ 2\ 2\ 4\ 4\ 3]'$; (c) a restricted domain with $(i, j) = (1, 3)$ and $\mathbf{N} = [0\ 4\ 0\ 4\ 1\ 1\ 5\ 2]'$.

3 Morphology of Restricted Domains

Let $A, B \subseteq \mathbf{Z}^2$. The *dilation* of A by B is denoted by $A \oplus B$ and is defined by $A \oplus B = \{c \in \mathbf{Z}^2 \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}$. The *n-fold dilation* of a set A by a set B is denoted by $A \oplus (\oplus_n B)$ and is defined as $A \oplus (\oplus_n B) = A \oplus B \oplus B \oplus \cdots \oplus B$ (n times).

In [1, 2] we gave algorithms for dilation, erosion, opening, closing, n-fold dilation and n-fold erosion of restricted domains using their boundary representations. We showed that the results obtained using these algorithms are equivalent to those obtained using conventional morphology and that the algorithms were constant-time. For the sake of completeness, in this section we give the results for dilation only.

Let A and B be two restricted domains with starting points (i_A, j_A) and (i_B, j_B) and side length vectors $\mathbf{N}^A$ and $\mathbf{N}^B$. The following lemma states that the dilation of A by B is just the addition of the respective side length vectors and the starting points. For proof see [1, 2].

Lemma 3.1 If $C = A \oplus B$, then $(i_C, j_C) = (i_A, j_A) + (i_B, j_B)$ and $\mathbf{N}^C = \mathbf{N}^A + \mathbf{N}^B$. Furthermore, if $D = A \oplus (\oplus_n B)$, then $(i_D, j_D) = (i_A, j_A) + n(i_B, j_B)$ and $\mathbf{N}^D = \mathbf{N}^A + n\mathbf{N}^B$. ∎

4 Decomposition of Restricted Domains

In this section, we interpret the problem of finding the decomposition as a vector space mapping problem.

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Figure 2: Example of $C = A \oplus B$. Here $(i_C, j_C) = (i_A, j_A) + (i_B, j_B) = (0,2) + (1,1) = (1,3)$; and $\mathbf{N}^C = \mathbf{N}^A + \mathbf{N}^B = [0\ 3\ 0\ 2\ 1\ 0\ 5\ 0]' + [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1]' = [0\ 4\ 0\ 3\ 1\ 1\ 5\ 1]'$.

4.1 A Basis Set

We claim that any restricted domain A can be decomposed as the k_i^{th}-fold dilations of thirteen basis structuring elements K_i shown in figure 3 as follows:

$$A = K_0 \oplus \left(\underset{k_1}{\oplus} K_1\right) \oplus \left(\underset{k_2}{\oplus} K_2\right) \oplus \cdots \oplus \left(\underset{k_{13}}{\oplus} K_{13}\right) \qquad (1)$$

where K_i is a member of $\mathcal{K}$, the basis set of structuring elements, and k_i are non-negative integers representing the number of times K_i is dilated. Notice that the K_i are either triangles, lines or rhombus. Thus, any restricted domain is a point in a thirteen dimensional space whose basis directions are thirteen shapes.

Figure 3: The thirteen basis sturcturing elements $K_1, \ldots, K_{13}$. Their corresponding $\mathbf{N}^{K_i}$ can be constructed as given in section 2.

Comparing the left hand side of equation (1) to its right hand side we see that for the above claim to be true, the lengths of the sides of the restricted domains on the L.H.S. and the R.H.S. should be the same. We can compute the dilations on the R.H.S., by using the dilation lemma (3.1), and finding the lengths of the sides of the resulting restricted domain in terms of k_i, $1 \leq i \leq 13$:

$$\mathbf{N}^A = \sum_{i=1}^{13} k_i \mathbf{N}^{K_i} \qquad (2)$$

where $\mathbf{N}^A$ and $\mathbf{N}^{K_i}$ are 8×1 vectors with the eight side lengths of the restricted domain A and the basis structuring elements K_i, respectively, and k_i are non-negative integers representing the k_i-fold dilation of K_i.

4.2 Linear Space Interpretation

From the previous section we see that solving the decomposition problem is equivalent to finding all the solutions of the set of equations (2). The set of linear equations in (2) can be rewritten using a matrix representation as follows:

$$\mathbf{T} \cdot \mathbf{K} = \mathbf{N}^A \qquad (3)$$

where $\mathbf{T} = [\mathbf{N}^{K_1}\mathbf{N}^{K_2} \cdots \mathbf{N}^{K_{13}}]$ is a 8×13 matrix whose columns $\mathbf{N}^{K_i}$ are 8×1 vectors representing the side lengths of the basis structuring elements K_i, $\mathbf{K}$ is a 13×1 vector, and $\mathbf{N}^A$ is the 8×1 vector with non negative integer entries representing the lengths of the sides of the restricted domain A. Thus, the decomposition problem can be restated as: given the matrix $\mathbf{T}$ and the vector $\mathbf{N}^A$, find a vector $\mathbf{K}$ such that its elements are non negative integers and such that it satisfies equation (3).

The matrix $\mathbf{T}$ is a linear transformation from a thirteen dimensional linear space into an eight dimensional linear space, that transforms the vector $\mathbf{K}$ into the vector $\mathbf{N}^A$. Therefore, to find the vector $\mathbf{K}$, we need to find the preimage of the vector $\mathbf{N}^A$ in the transformation $\mathbf{T}$. That is, we need to find all the vectors $\mathbf{K}$ whose image under the transformation $\mathbf{T}$ is $\mathbf{N}^A$.

Since the dimension of the null space, $\mathbf{K}^h$, of the transformation $\mathbf{T}$ is not zero [1, 3], the vector $\mathbf{N}^A$ has more than one preimage in $\mathbf{T}$ and there is more than one possible decomposition of the restricted domain. In general, we can express any preimage of $\mathbf{N}^A$ as the sum of one particular preimage $\mathbf{K}^p$ and a vector from the null space $\mathbf{K}^h$. Finally, all the solutions, $\mathbf{K}^h$, can be expressed as $\mathbf{K}^g = \mathbf{K}^p + \mathbf{K}^h$. This is summarized in figure 4.

Figure 4: $\mathbf{T}$ is a linear transformation from a 13 dimensional space, in which the restricted domains are represented in terms of dilation decompositions, $\mathbf{K}$, to a 8 dimensional space, in which the restricted domains are represented in terms of their side lengths, $\mathbf{N}$.

4.2.1 A Particular Solution

A particular solution of the system of equations (3), $\mathbf{K}^p$, can be found by looking at the underlying geometry of the problem. Note that finding a particular solution to equation (3) simultaneously proves the existence of the decomposition of a restricted domain. The algorithm for finding a particular solution is given in [1, 3, 4, 5]. There, it was shown that the algorithm for the particular solution of the decomposition of a restricted domain consists of only assignment statements, comparisons, and no loops and thus is finite-time.

4.2.2 The Homogeneous Solution

The homogeneous equation associated with equation (3) is given by:

$$\mathbf{T} \cdot \mathbf{K} = 0 \ . \qquad (4)$$

The solution $\mathbf{K}^h$ of the above equation is the null subspace of the space spanned by the rows of the matrix $\mathbf{T}$, or in other words, the kernel of the transformation $\mathbf{T}$. Since the rank of the matrix $\mathbf{T}$ is six [1, 3], the transformation $\mathbf{T}$ is a mapping of a thirteen dimensional space onto a six dimensional space with a seven dimensional null space. Thus, we can find a set of basis vectors $\mathbf{v}_j$, $1 \le j \le 7$, which span the null space of $\mathbf{T}$. For details on the construction of these basis vectors see [1, 3]. The homogeneous solution $\mathbf{K}^h$ of $\mathbf{T}$ are, then, all the vectors expressible as the linear combination of the basis vectors of the null space. That is,

$$\mathbf{K}^h = \sum_{i=1}^{7} \alpha_i \mathbf{v}_i \ , \ \alpha_i \in \mathbf{R} \ . \qquad (5)$$

4.2.3 The General Solution

From linear spaces, we know that the general solution to equation (3) is the sum of a particular solution and the homogeneous solution. That is,

$$\mathbf{K}^g = \mathbf{K}^p + \mathbf{K}^h = \mathbf{K}^p + \sum_{i=1}^{7} \alpha_i \mathbf{v}_i \ , \ \alpha_i \in \mathbf{R} \ . \qquad (6)$$

But we know that the general solution can have only non negative entries since they represent the n-fold dilation of the structuring elements. Furthermore, the entries of the particular solution $\mathbf{K}^p$ are non negative integers. Thus, the α_i's must belong to $\mathbf{Z}$, [1, 3]. We notice that equation (2) provides an upper and lower bound for the k_i's since the R.H.S. are additions of non negative integers, k_i, which cannot exceed the non negative integer constants on the L.H.S. The bounds obtained can be written compactly using matrices as $\mathbf{0} \le \mathbf{K}^g \le \mathbf{K}^{max}$ where $\mathbf{K}^{max}$ is a vector having the thirteen upper bounds as its elements, [1, 3]. Using equation (6) and the bounds on $\mathbf{K}^g$, we can further constraint the space of $\boldsymbol{\alpha}$, [1, 3]: $-\mathbf{K}^p_{6+i} \le \alpha_i \le \mathbf{K}^{max}_{6+i} - \mathbf{K}^p_{6+i}$ for $1 \le i \le 7$ and $\alpha_i \in \mathbf{Z}$. Thus, the general solution $\mathbf{K}^g$ can be found by substituting the α_i values from the valid domain defined above in equation (6). However, it has to be verified that the obtained $\mathbf{K}^g$ is within bounds. This has to be done for all points in the $\boldsymbol{\alpha}$ space. The algorithm for generating *all* the solutions of the decomposition problem is given in table 1.

The complexity of the algorithm to generate all the possible decompositions of a restricted domain (table 1) is a function of the size of the restricted domain. The complete $\boldsymbol{\alpha}$ domain has to be searched for all the legal $\mathbf{K}^g$s. Thus, the complexity of the algorithm is of the order of the number of $\boldsymbol{\alpha}$ vectors, i.e., complexity $= O((\mathbf{K}^{max}_7) \times (\mathbf{K}^{max}_8) \times \cdots \times (\mathbf{K}^{max}_{13}))$.

Table 1: Algorithm for the general solution.

procedure DecomposeGeneral($\mathbf{N}^A$, $\mathbf{K}^p$, $\mathbf{K}^g$)
Input: $\mathbf{N}^A$ (side lengths of A), $\mathbf{K}^p$ (a particular soln.);
Output: $\mathbf{K}^g$ (the set of all decompositions);

begin
  Initialize $\mathbf{K}^{max}$ using $\mathbf{N}^A$;
  Initialize bounds on α_i: $-\mathbf{K}^p_{6+i} \le \alpha_i \le \mathbf{K}^{max}_{6+i} - \mathbf{K}^p_{6+i}$;
  for each $\alpha \in \mathbf{Z}^7$ *and* within bound **do**
    Construct $\mathbf{K} = \mathbf{K}^p + \sum_{i=1}^{7} \alpha_i \mathbf{v}_i$;
    if $\mathbf{0} \le \mathbf{K} \le \mathbf{K}^{max}$
      Add $\mathbf{K}$ to the set $\mathbf{K}^g$;
  end For ;
end DecomposeGeneral;

5 Conclusion

We defined restricted domains – a restricted class of 2-D shapes. We proved that any restricted domain can be decomposed as n-fold dilations of thirteen basis structuring elements and hence can be represented in a thirteen dimensional space. This thirteen dimensional space is spanned by the thirteen basis structuring elements comprising of lines, triangles, and a rhombus. We showed that there is a linear transformation from this thirteen dimensional space to an eight dimensional space wherein a restricted domain is represented in terms of its side lengths. Furthermore, we showed that the decomposition in general is not unique, and all the decompositions can be constructed by finding the homogeneous solutions of the transformation and adding it to a particular solution. An algorithm for finding all possible decompositions was provided.

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